THE FLOW OF A VISCOUS LIQUID PAST A DOUBLY PERIODIC ROW OF CYLINDERS

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Abstract: The flow of a liquid in a system of parallel cylinders arranged perpendicularly to the flow is considered for small Reynolds numbers.

Comparison of the exact solution for square and hexagonal arrays with the solution obtained by means of the cell model established that in the case of an array the method proposed by Kuvabara [1] makes it the flow velocity over a single cylinder to be correctly calculated. However, the form of the streamlines depends essentially on the geometry of the array.

**1. The cell model and the Oseen solution.** In [1,2], the problem of the flow of liquid around a finite system of spatially uniformly arranged cylinders is considered by means of the so-called cell model. If each of the cylinders of radius $a$ is surrounded by an imaginary coaxial cylinder whose radius b is given by the relation $\pi b^2 n = 1$ (n is the number of cylinders per unit cross-sectional area), it is assumed that the choice of the simplest, physically plausible boundary conditions at the surface of the outer cylinder does not significantly alter the velocity field close to the inner cylinder. The groundlessness of such an assumption is obvious even in the comparison of the results from [1,2].

In Kuabarsa' [1], in a reference system where the liquid at infinity is at rest, the vortex, as well as the radial component of the velocity of motion of the liquid, vanish on the surface of a cylinder of radius b. The last assumption means that the streamlines far from the cylinder approximate circles. Concerning the first assumption, we must point out that, for example, it is valid for a hexagonal array only at six points on a circle of a radius equal to the half-period of the array. This can be easily seen from considerations of symmetry.

The solution of the Stokes equations in dimensionless polar coordinates r, θ, with the origin of the coordinates on the axis of the cylinder and the boundary conditions for the velocity components

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{\partial\psi}{\partial r},$$

$$v_r = v_\theta = 0 \ \text{for}\ r = a, \qquad v_r = U\cos\theta \ \text{for}\ r = b,$$

$$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{\partial v_r}{\partial\theta} = 0 \qquad \text{for}\ r = b \tag{1.1}$$

leads to the following expression for the stream function:

$$\psi(r,\theta) = \frac{Ua\sin\theta}{2}\left(-\frac{\ln\varepsilon}{2} + \varepsilon - \frac{\varepsilon^2}{4} - \frac{3}{4}\right)^{-1}\times$$
$$\times\left[\left(1-\frac{\varepsilon}{2}\right)\frac{a}{r} - (1-\varepsilon)\frac{r}{a} + 2\frac{r}{a}\ln\frac{r}{a} - \frac{\varepsilon r^3}{2a^3}\right], \tag{1.2}$$

where $\varepsilon = a^2/b^2$ is the fraction of the volume occupied by cylinders (r is the distance to the axis of the cylinder; θ is the angle between the radius vector r and U).*

Happel [2] assumes that in a reference system where the liquid at infinity is at rest, the radial velocity component and the component of the tensor of viscous stresses are both zero on the surface of a cylinder of radius b. Thus, in a reference system connected with the axis of some cylinder, the following boundary conditions

$$v_r = v_\theta = 0 \ \text{for}\ r = a \ \text{and}\ v_r = U\cos\theta \ \text{for}\ r = b \tag{1.3}$$

must be satisfied.

* Because of a misprint formula (1.2) in [1] is represented in a distorted form which was corrected in [3].

In this case, the stream function

$$\psi(r,\theta) = \frac{Ua\sin\theta}{2}\left[-\frac{\ln\varepsilon}{2} + \frac{\varepsilon^2}{2(1+\varepsilon^2)} - \frac{1}{2}\right]^{-1}\times$$
$$\times\left[\frac{a}{(1+\varepsilon^2)r} - \frac{(1-\varepsilon^2)r}{(1+\varepsilon^2)a} + 2\frac{r}{a}\ln\frac{r}{a} - \frac{\varepsilon^2 r^3}{(1+\varepsilon^2)a^3}\right].$$

Only in the limiting case as $\varepsilon \to 0$ do formulas (1.2), (1.4), and the Oseen solution for the problem [4] of flow around a single cylinder

$$\psi(r,\theta) = \frac{Ua\sin\theta}{2}\left(\ln\frac{\gamma}{2}ka - \frac{1}{2}\right)^{-1}\left(\frac{a}{r} - \frac{r}{a} + 2\frac{r}{a}\ln\frac{r}{a}\right) \tag{1.5}$$

leads to the same streamlines ($\gamma = 1.78$; $k = U|2\nu$; $\nu$ is the kinematic viscosity.)

Thus, the question concerning the choice of boundary conditions in the cell method, purporting to give a correct description of the velocity field close to the surface of the cylinder (although with accuracty to terms of order ε), still remains open. In this connection, it is interesting to investigate the exact solution of the Stokes equations for a doubly periodic region. In the paper by Hasimoto [5], the problem concerning the flow of a viscous liquid through a doubly periodic system of cylinders passing through the nodes of a square array, is considered with the application of Fourier series. We calculate force acting on a cylinder when ε is small. But no expression is presented in [5], for the velocity field close to the vicinity of the cylinder. This is necessary, for example, for the calculation of the diffusion flow on the surface of the cylinder.

The problem of the flow of a viscous liquid in a doubly periodic system of cylinders is solved here by means of the theory of elliptic functions [6]. The axes of the cylinders pass through the nodes of a two-dimensional array; the periods of this array can be expressed by the numbers $2\omega_1$ and $2\omega_2$ in the complex plane, where $\omega_1 = \bar{\omega}_1$ and $\omega_2 = |\omega_2|e^{i\varphi}$ (the bar denotes the operation of complex conjugation). The calculation of the velocity field for $\omega_1 = |\omega_2|$ is carried out for the two particular cases: $\varphi = \pi 1/2$ and $\varphi = \pi 1/3$. Here it is assumed that the liquid flow velocity U, averaged across the section, is parallel to one of the periods of the array.

**2. The Stokes equations and their general solution.** In a plane perpendicular to the axis of the cylinders we choose Cartesian (x,y) coordinates so that the origin coincides with the center of some cylinder, while the (real) x axis coincides with the flow direction.

Let $v_x$ and $v_y$ be the x and y components of the liquid flow velocity vector; let μ be the dynamic viscosity, and let p be the pressure.

The two-dimensional Stokes equations and the equation of continuity are written in the form

$$\frac{\partial p}{\partial x} = \mu\Delta v_x, \qquad \frac{\partial p}{\partial y} = \mu\Delta v_y,$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{2.1}$$

On the surface of each cylinder,

$$v_x = v_y = 0. \tag{2.2}$$

The rate of flow of liquid through the cell is assumed to be given:

$$\int_a^{|\omega_2|\sin\varphi} v_x(0,y)\,dy = U|\omega_2|\sin\varphi. \tag{2.3}$$

Eliminating the pressure p from Eqs. (2.1), we obtain

$$\Delta\omega_0 = 0, \quad \omega_0 = \partial v_y/\partial x - \partial v_x/\partial y, \tag{2.4}$$

where $\omega_0$ is the vorticity of the liquid, which in the case under consideration is a real function of the coordinates (x,y) as well as of the parameters U, $\omega_1$, and $|\omega_2|$. In conformity with the method of functions of the complex variable (see [7], for example), we go over to the new variables $z = x + iy$ $\bar{z} = x - iy$ and introduce the complex velocity $w = v_x - iv_y$. Then Eqs. (2.4) are written in the form

$$\partial^2\omega_0/\partial z\,\partial\bar{z} = 0, \qquad \omega_0 = 2i\,\partial w/\partial\bar{z}. \tag{2.5}$$

An analogous method was used by Miyagi [8] in considering the problem of viscous flow through an infinite, singly periodic array of cylinders, whose axes lie in a plane parallel to the velocity U of the impinging flow. From Eq. (2.5) it follows that $\omega_0 = G(z) + F(\bar{z})$, where G(z) and F($\bar{z}$) are analytical functions of z and $\bar{z}$, respectively. Since $\omega_0 = \bar{\omega}_0$, where $F(\bar{z}) = \bar{G}(\bar{z}) = \bar{\Omega}(\bar{z})$,

$$\omega_0 = \Omega(z) + \bar{\Omega}(\bar{z}). \tag{2.6}$$

The velocity field is symmetric with respect to substitution of z and $\bar{z}$ by $-z$ and $-\bar{z}$. Therefore, $\omega_0(z, \bar{z}) = -\omega_0(-z, -\bar{z})$ and, hence, $\Omega(z) = -\Omega(-z)$. From formula (1.5) we can see that the Laurent expansion for $\omega_0$, with small r, begins with terms of order $a/r$. Then, the doubly periodic function $\omega_0$ is sought in the form of a series containing the Weierstrass $\zeta$-function and its even-order derivatives. We write $\Omega(z)$ in the form

$$\Omega(z) = \frac{U}{i}\left\{a_0\left[\zeta(z) - \frac{z(\eta_2 - \bar{\eta}_2)}{\omega_2 - \bar{\omega}_2}\right] + \sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k)}(z)\right\}, \tag{2.7}$$

where $\zeta(z, 2\omega_1, 2\omega_2)$ is a $\zeta$-functions with the period $2\omega_1$, $2\omega_2$, $\eta_i = \zeta(\omega_i)$ ($i = 1, 2$). In the region $z \ll \max(\omega_1, |\omega_2|)$ the $\zeta$-function can be represented in the form of the series

$$\zeta(z;\ 2\omega_1,\ 2\omega_2) = 1/z - g_2 z^3 - g_3 z^5 - \ldots,$$

$$g_2 = \sum_{m,n}{}' \Omega_{mn}^{-4}, \qquad g_3 = \sum_{m,n}{}' \Omega_{mn}^{-6}.$$

Here $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$; summation is extended to all integer values of m and n, with the exception of $m = n = 0$

By virtue of the symmetry of the array relative to the plane $y = 0$, the quantities $g_2$ and $g_3$ are real, and consequently $\bar{\zeta}(z) = \zeta(z)$. Furthermore, from the same considerations, it follows that the coefficients $a_0$ and $a_{2k}$ are real, since

$$\omega_0(z, \bar{z}) = -\omega_0(\bar{z}, z),\ \Omega(z) = -\bar{\Omega}(z).$$

Thus

$$\omega_0(z, \bar{z}) = \frac{U}{i}\left\{a_0\left[\zeta(z) - \zeta(\bar{z}) - \frac{\eta_2 - \bar{\eta}_2}{\omega_2 - \bar{\omega}_2}(z - \bar{z}) + \right.\right.$$

$$\left. + \sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k)}(z) - \sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k)}(\bar{z})\right\}. \tag{2.8}$$

From (2.8), with (2.5) taken into account, we find that

$$\frac{2w}{U} = a_0\left\{\ln\sigma(z) - \bar{z}\zeta(z) - \frac{\eta_2 - \bar{\eta}_2}{\omega_2 - \bar{\omega}_2}\left(\frac{\bar{z}^2}{2} - z\bar{z}\right) - \right.$$

$$\left. - \bar{z}\sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k)}(z) + \sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k-1)}(\bar{z}) + \Phi(z)\right\}, \tag{2.9}$$

where $\sigma(z; 2\omega_1, 2\omega_2)$ is the Weierstrass sigma-function with the periods $2\omega_1$ and $2\omega_2$, given by the equation $(d/dz) \ln \sigma(z) = \zeta(z)$, while $\Phi(z)$ is an arbitrary analytic function of z. However, certain restrictions associated with the double periodicity and limitations of the velocity field must be imposed on our arbitrary choice of $\Phi(z)$. To find a suitable $\Phi(z)$ we differentiate (2.9) with respect to z. Then,

$$\chi(z, \bar{z}) = -\bar{z}\Delta(z) + \Phi'(z), \qquad \chi(z, \bar{z}) = \frac{2}{U}\frac{\partial w}{\partial z},$$

$$\Delta(z) = a_0\zeta'(z) - a_0\frac{\eta_2 - \bar{\eta}_2}{\omega_2 - \bar{\omega}_2} + f'(z),$$

$$f(z) = \sum_{k=1}^{\infty} a'_{2k}\zeta^{(2k)}(z).$$

Since

$$\chi(z + 2\omega_i,\ \bar{z} + 2\bar{\omega}_i) = \chi(z, \bar{z}) \qquad (i = 1, 2),$$

then

$$\Phi'(z + 2\omega_1) - \Phi'(z) =$$

$$= 2\omega_1\Delta(z),\ \Phi'(z + 2\omega_2) - \Phi'(z) = 2\bar{\omega}_2\Delta(z).$$

Hence

$$\Phi'(z) = \left\{z + \frac{2}{\pi i}(\omega_2 - \bar{\omega}_2)[\omega_1\zeta(z) - \eta_1 z]\right\}\Delta(z) +$$

$$+ \sum_{k=1}^{\infty} D_{2k}\zeta^{(2k)}(z). \tag{2.10}$$

Here $D_{2k}$ are constants. It is easy to show that $\Phi(z)$ chosen in such a manner is unique, and accurate to an arbitrary odd elliptic function having a single pole of any order above the second at points comparable with $z = 0$.

Integrating (2.10), we obtain

$$\Phi(z) = a_0[\ln\sigma(z) + \alpha\zeta^2(z) + \beta z\zeta(z) - \beta\gamma z^2] +$$

$$+ \beta z f(z) + \sum_{k=1}^{\infty} b_{2k}\zeta^{(2k-1)}(z) + 2\alpha\int\zeta(z)f'(z)\,dz + K,$$

$$\alpha = \frac{\omega_1}{\pi i}(\omega_2 - \bar{\omega}_2), \qquad \beta = 1 - \frac{2\eta_1}{\pi i}(\omega_2 - \bar{\omega}_2),$$

$$\gamma = \frac{\eta_2 - \bar{\eta}_2}{2(\omega_2 - \bar{\omega}_2)}. \tag{2.11}$$

Here $b_{2k}$ and K are constants.

Thus, the solution of the Stokes equations (2.1) can be sought in the form

$$\frac{2w}{U} = a_0\{\ln\sigma(\bar{z}) + \ln\sigma(z) + [\beta z - \bar{z} + \alpha\zeta(z)]\zeta(z) -$$

$$- \gamma(\beta z^2 - 2z\bar{z} + \bar{z}^2)\} +$$

$$+ (\beta z - \bar{z})\sum_{k=1}^{\infty} a_{2k}'\zeta^{(2k)}(z) + \sum_{k=1}^{\infty} a_{2k}'\zeta^{(2k-1)}(\bar{z}) +$$

$$+ \sum_{k=1}^{\infty} c_{2k}'\zeta^{(2k-1)}(z) + 2\alpha\sum_{k=1}^{\infty} a_{2k}'\zeta(z)\zeta^{(2k)}(z) + K.$$

The coefficients $a_{2k}'$, $c_{2k}'$, and K are determined from the boundary conditions (2.2) and (2.3).

For the case in which $\bar{\omega}_2 = -\omega_2$, we can obtain an asymptotic expression of the general solution (2.12) where $\omega_1 \gg |\omega_2|$. The result thus obtained corresponds to the general form of the solution for the singly periodic array of cylinders given in [8].

**3. Square and hexagonal arrays.** Let $|\omega_2| = \omega_1 = \omega$, i.e., $\omega_2 = \omega e^{i\varphi}$. The symmetry requirement of the velocity field relative to the real axis leads to the fact that only the values $\varphi = \pi 2$ and $\varphi = \pi 3$ are permissible, i.e., when they correspond to arrays with elemental cells in the form of a square or a rhombus with an angle of 60 degrees. In these cases, directly from the definition of the $\zeta$-function, we can derive the relations

$$\eta_1 = \bar{\eta}_1, \qquad \eta_2 = \bar{\eta}_1 e^{-i\varphi} = \eta_1 e^{i\varphi}. \tag{3.1}$$

Since $\eta_1\omega e^{i\varphi} - \eta_2\omega = \pi/2$, from (3.1) we obtain

$$\eta_1 = \frac{\pi}{4\omega\sin\varphi}, \qquad \alpha = \frac{2\omega^2}{\pi}\sin\varphi,$$

$$\beta = 0, \qquad \gamma = -\frac{\pi}{8\omega^2 \sin\varphi}. \tag{3.2}$$

With this result taken into consideration, we write expression (2.12) as follows:

$$\frac{2w}{U} = a_0 \Big[\ln\sigma(\bar{u}) + \ln\sigma(u) + \frac{\sin\varphi}{2\pi}\zeta^2(u) - \bar{u}\zeta(u) + \frac{\pi}{2\sin\varphi}(\bar{u} - 2u)\bar{u}\Big] - \bar{u}\sum_{k=1}^{\infty} a_{2k}\zeta^{(2k)}(u) + \sum_{k=1}^{\infty} a_{2k}\zeta^{(2k-1)}(\bar{u}) + \sum_{k=1}^{\infty} c_{2k}\zeta^{(2k-1)}(u) + \frac{\sin\varphi}{\pi}\sum_{k=1}^{\infty} a_{2k}\zeta(u)\zeta^{(2k)}(u) + K. \tag{3.3}$$

Elliptic functions of the new variable $u = z/2\omega$, allowing the transformation from functions with period $2\omega, 2\omega e^{i\varphi}$ to functions with period $1, e^{i\varphi}$, enter (3.3):

$$\zeta(z;\,2\omega,\,2\omega e^{i\varphi}) = \frac{1}{2\omega}\zeta(u;\,1,\,e^{i\varphi}),$$

$$\ln\sigma(z;\,2\omega,\,2\omega e^{i\varphi}) = \ln\sigma(u;\,1,\,e^{i\varphi}).$$

The constants in formulas (2.12) and (3.3) are associated with the relations

$$a_{2k} = a_{2k}'/2(\omega)^{2k} \text{ and } c_{2k} = c_{2k}'/(2\omega)^{2k}.$$

Owing to the double periodicity of the velocity field, boundary conditions (2.2) are adequately specified on the surface of any single cylinder: $w = 0$ for $|z| = a$ or $z\bar{z} = a^2$, i.e., for $\bar{u} = t^2|u(t = a|2\omega)$.

Substituting the known Laurent expansions of elliptical functions into (3.3) and equating the coefficients with the same powers of u, for $a_0, a_{2k}, c_{2k}$, and K we obtain an infinite system of equations. In the general case (2.12), the calculation of coefficients represents a fairly laborious problem. But, for square and hexagonal arrays, the calculations are considerably simplified, since for the former $g_3 = 0$, and for the latter $g_2 = 0$ (see Appendix 1). To satisfy the boundary conditions with accuracy to second order $\varepsilon = \pi t^2$, in the case of a square array, we must determine the following coefficients:

$$\frac{a_2}{a_0} = -\frac{t^4 g_2}{2}\left(\frac{5}{\pi} - 8t^2\right), \qquad \frac{a_4}{a_0} = -\frac{t^8 g_2}{24},$$

$$\frac{c_2}{a_0} = \frac{1}{2\pi} - t^2 + \frac{\pi t^4}{2}, \qquad \frac{c_4}{a_0} = -\frac{t^4 g_2}{6\pi}\left(\frac{5}{\pi} - 13t^2\right),$$

$$\frac{c_6}{a_0} = -\frac{t^8 g_2}{120\pi},$$

$$\frac{K}{a_0} = -2\ln t + \pi t^2\frac{5t^4 g_2}{\pi} - \left(\frac{5}{\pi} - 12t^2\right),$$

$$\left(g_2 = \sum_{m,n}{}' (m + in)^{-4}\right). \tag{3.4}$$

From the relations

$$v_x = \frac{\partial\psi}{\partial y}, \qquad v_y = -\frac{\partial\psi}{\partial x}$$

giving the stream function, it is not difficult to obtain

$$\psi(u,\bar{u}) = \frac{\omega}{i}\int w(u,\bar{u})\,du + \Psi(\bar{u}), \tag{3.5}$$

we choose $\Psi(\bar{u})$ so that $\psi(u,\bar{u})$ are real:

$$\psi = Ua_0\omega\,\mathrm{Im}\Big\{-\bar{u}\ln\sigma(u) + \int\ln\sigma(u)\,du + \frac{1}{2\pi}\int\zeta^2(u)\,du + \frac{\pi}{2}u\bar{u}^2 - \bar{u}\sum_{k=1}^{\infty}\frac{a_{2k}}{a_0}\zeta^{(2k-1)}(u) + \sum_{k=1}^{\infty}\frac{c_{2k}}{a_0}\zeta^{(2k-2)}(u) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{a_{2k}}{a_0}\int\zeta(u)\zeta^{(2k)}(u)\,du + \frac{K}{a_0}u\Big\}. \tag{3.6}$$

The constant $a_0$ can be determined from the condition

$$\psi(1/2i, -1/2i) - \psi(it, -it) = \omega U. \tag{3.7}$$

Hence, it follows that, with accuracy to terms of order ε (see Appendix 2),

$$\frac{1}{a_0} = -\frac{\ln\varepsilon}{2} + \varepsilon - \lambda$$

$$\lambda = -\frac{1}{2}\ln\sigma\left(\frac{1}{2}\right) + \frac{3}{4} - \frac{1}{\pi} + \frac{\pi}{16} - \frac{g_2}{24\pi} + \frac{g_2}{640} = 0.739. \tag{3.8}$$

We find the expression for the stream function close to the surface of the cylinder, with accuracy to terms of order ε. For this, we restrict ourselves to the first terms of the Laurent series for functions entering (3.6) and substitute $u = re^{i\theta}/2\omega$ into the resulting expression.

Thus, we arrive at the following result:

$$\psi(r,\theta) = \frac{aU\sin\theta}{2(-1/2\ln\varepsilon + \varepsilon - \lambda)} \times \left[2\frac{r}{a}\ln\frac{r}{a} + \left(1 - \frac{\varepsilon}{2}\right)\frac{a}{r} - (1-\varepsilon)\frac{r}{a} - \frac{\varepsilon r^3}{2a^3} - \varepsilon\frac{\sin 3\theta}{\sin\theta}\frac{5g_2}{6\pi^2}\left(2\frac{a^3}{r^3} - 3\frac{a}{r} + \frac{r^3}{a^3}\right)\right]. \tag{3.9}$$

Let us try to determine how the geometry of the array influences the velocity of the impinging flow and the velocity field close to the surface of the cylinder. With this aim we carry out an analogous calculation for the hexagonal array. In Eq. (3.3) we set $\varphi = 1\pi/3$ and by the method mentioned above we find the unknown coefficients $a_{2k}$ and $b_{2k}$. To satisfy the boundary conditions with accuracy to terms of order $\varepsilon^2$, we must determine

$$\frac{K}{a_0} = \frac{2\pi t^2}{\sqrt{3}} - 2\ln t + o(t^8),$$

$$\frac{c_2}{a_0} = \frac{\sqrt{3}}{4\pi} - t^2 + \frac{\pi}{\sqrt{3}}t^4 + o(t^8)$$

$$\frac{a_4}{a_0} = t^8 g_3\left(t^2 - \frac{7\sqrt{3}}{24\pi}\right), \qquad \frac{a_6}{a_0} = -\frac{t^{12}g_3}{6!},$$

$$\frac{c_6}{a_0} = \frac{t^8 g_3\sqrt{3}}{120\pi}\left(19t^2 - \frac{7\sqrt{3}}{2\pi}\right), \qquad \frac{c_8}{a_0} = -\frac{t^{12}g_3\sqrt{3}}{7!\,2\pi},$$

$$a_2 = 0, \qquad c_4 = 0. \tag{3.10}$$

To find $a_0$ we use the condition

$$\psi(1/2\,e^{i\varphi},\ 1/2\,e^{-i\varphi}) - \psi(it, -it) = \omega U\sin\varphi. \tag{3.11}$$

Here, in conformity with (3.5), for $\psi$ we have the expression

$$\psi = Ua_0\,\mathrm{Im}\Big\{-\bar{u}\ln\sigma(u) + \int\ln\sigma(u)\,du + \frac{\sqrt{3}}{4\pi}\int\zeta^2(u)\,du + \frac{\pi}{\sqrt{3}}u\bar{u}^2 - \bar{u}\sum_{k=1}^{\infty}\frac{a_{2k}}{a_0}\zeta^{(2k-1)}(u) + \sum_{k=1}^{\infty}\frac{c_{2k}}{a_0}\zeta^{(2k-2)}(u) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{a_{2k}}{a_0}\int\zeta(u)\zeta^{(2k)}(u)\,du + \frac{K}{a_0}u\Big\}. \tag{3.12}$$

We obtain (see Appendix 2)

$$1/a_0 = -1/2\ln\varepsilon + \varepsilon - 1/8\sqrt{3}\varepsilon^2 - \lambda$$

$$\varepsilon = \frac{2\pi t^2}{\sqrt{3}},$$

$$\lambda = \frac{3}{4} + \frac{1}{8\sqrt{3}} - \frac{\sqrt{3}}{2\pi} - \frac{g_3\sqrt{3}}{280\pi} - \frac{\pi}{6\sqrt{3}} +$$

$$+ \frac{1}{2}\ln\sqrt{3} - \frac{1}{2}\ln\pi +$$

$$+ \frac{1}{\sqrt{3}}\,\mathrm{Im}\left\{e^{-1/3 i\pi}\ln\sigma\left(\frac{1}{2}e^{1/3 i\pi}\right)\right\} = 0.754. \qquad (3.13)$$

Thus,

$$\psi(r,\theta) = \frac{Ua\sin\theta}{2(-1/2\ln\varepsilon + \varepsilon - 1/8\sqrt{3}\varepsilon^2 - \lambda)} \times$$

$$\times\left\{2\frac{r}{a}\ln\frac{r}{a} + \left(1 - \frac{\varepsilon}{2}\right)\frac{a}{r} - (1-\varepsilon)\frac{r}{a} - \frac{\varepsilon r^3}{2a^3} -\right.$$

$$\left. - 0.128\varepsilon^2\left[\frac{4\sin 3\theta}{\sin\theta}\left(\frac{a^5}{r^5} - \frac{a^3}{r^3}\right) + \frac{3\sin 5\theta}{5\sin\theta}\left(\frac{r^5}{a_5} - \frac{a^3}{r^3}\right)\right]\right\}. \qquad (3.14)$$

Comparing (3.9) and (3.14) with (1.2) it should be noted that the method of Kuvabara [1] enables us to determine the velocity of the impinging flow with accuracy to terms of order $\varepsilon$. Apparently, this quantity, with the accuracy just mentioned, depends only slightly on the geometry of the array. This can easily be seen by comparing formulas (3.9) and (3.14).

As for the streamlines on the basis of the results obtained here, it is not difficult to establish that they depend essentially on the mutual arrangement of the cylinders. In this connection, attention should be paid to the considerable divergence, of the stream function for a square array from the Kuwabara formula in the case of a hexagonal array, a completely satisfactory agreement, with accuracy to terms of order $\varepsilon$, is observed.

It is easy to see that both the velocity of the impinging flow and the streamlines found by Happel differe considerably from those obtained were to. This indicates an unsuccessful choice of boundary conditions in the cell method proposed by Happel. This is also confirmed by the experimental data obtained by A. A. Kirsh and N. A. Fuks, which lead to the value $\gamma = 0.75 + 0.02$.

As we know, the force acting on the cylinder is given [4] by the formula

$$F_x + iF_y = \oint (ip + \mu\omega_0)\,dz,$$

where $F_x$ and $F_y$ are the x and y components of the force, and the integration is carried out over the surface of the cylinder. In the Stokes approximation [8] we have $ip + \mu\omega = 2\mu\,\Omega(z)$, and consequently,

$$F_x + iF_y = 4\pi\mu U a_0. \qquad (3.15)$$

Substituting the value $a_0$ from (3.8) and (3.13) into (3.15), we obtain

$$F_x = \frac{4\pi\mu U}{-1/2\ln\varepsilon + \varepsilon - \lambda}, \qquad (3.16)$$

where $\lambda = 0.739$ for a square array, and $\lambda = 0.754$ for a hexagonal array.

Expression (3.16), for the case of a square array, does not differ from the formula of Hasimoto [5], while for a hexagonal array it virtually coincides with the result of Kuvabara [1].

**Appendix 1. Calculation of $g_2$ and $g_3$ for hexagonal and square arrays.** We use the known relations

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_1e_3 + e_2e_3 = -1/4\, g_2^0,$$

$$e_1 e_2 e_3 = 1/4\, g_3^0, \quad e_1 = \rho(\omega_1), \quad e_2 = \rho(\omega_2),$$

$$e_3 = \rho(-\omega_1 - \omega_2),$$

$$g_2^0 = 60\sum_{m,n}{}'(2m\omega_1 + 2n\omega_2)^{-4} = 60g_2$$

$$g_3^0 = 140\sum_{m,n}{}'(2m\omega_1 + 2n\omega_2)^{-6} = 140g_3.$$

From the definition of $\rho(u)$, it follows directly that

$$\varphi = 1/2\pi,\ 1/3\pi, \quad e_1 = \bar{e}_2 e^{-2i\varphi},$$

$$e_3 = -e_1(1 + e^{-2i\varphi}), \quad 1/4\, g_2 = e_1^2(e^{-3i\varphi}\,2\cos\varphi + 1),$$

$$1/4\, g_3 = -e_1^3 e^{-3i\varphi}\,2\cos\varphi.$$

Thus, $g_3 = 0$ and $g_2 = 4e_1^2$ for $\varphi = \pi/2$; $g_2 = 0$ and $g_3 = 4e_1^3$ for $\varphi = \pi/3$.

Using the expression for $e_1$ in terms of the theta-functions [9],

$$e_1 = 1/12\,\pi^2\omega^{-2}(\vartheta_4^4 + \vartheta_3^4),$$

for the case of a square array we obtain the value $e_1 = 0.697\pi^2$, while for a hexagonal array we have $e_1 = 0.598\pi^2$.

Consequently, for a square array, $g_2^0 = 1.944\,\pi^4$; for a hexagonal array, $g_3^0 = 0.854\,\pi^6$.

**Appendix 2. Calculation of $\ln\sigma(i/2)$ and $\ln\sigma(1/2\, e^{1/3 i\pi})$.** From the definition of $\sigma(z, 2\omega_1, 2\omega_2)$, we obtain

$$\ln\sigma(1/2\, e^{i\varphi};\ 1,\ e^{i\varphi}) = i\varphi + \ln\sigma(1/2).$$

Using the expression for $\sigma(z)$ in terms of the theta-functions [6], we find

$$\ln\sigma(1/2;\ 1,\ i) = 1/8\,\pi - \ln\pi + 0.008$$

$$\ln\sigma(1/2;\ 1,\ e^{1/3 i\pi}) = 1/8\,\pi - \ln\pi - 0.017.$$

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